Asians and cash dividends: Exploiting symmetries in pricing theory

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Abstract

In this article we present new results for the pricing of arithmetic Asian options within a Black-Scholes context. To derive these results we make extensive use of the local scale invariance that exists in the theory of contingent claim pricing. This allows us to derive, in a natural way, a simple PDE for the price of arithmetic Asians options. In the case of European average strike options, a proper choice of numeraire reduces the dimension of this PDE to one, leading to a PDE similar to the one derived by Rogers and Shi. We solve this PDE, finding a Laplace-transform representation for the price of average strike options, both seasoned and unseasoned. This extends the results of Geman and Yor, who discussed the case of average price options. Next we use symmetry arguments to show that prices of average strike and average price options can be expressed in terms of each other. Finally we show, again using symmetries, that plain vanilla options on stocks paying known cash dividends are closely related to arithmetic Asians, so that all the new techniques can be directly applied to this case.

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1 Introduction.

Options with a payoff-function which depends on the average of some underlying, a.k.a. Asian-type option have a multitude of applications in finance. They find applications for example in currency-based contracts, interest rates and commodities. In the following we will consider a setting where stock prices are modeled by geometric Brownian motions. Depending on the type of averaging the analytic price of such a contract is easy or difficult to compute. Geometric averaging leads to simple expressions for the prices [HN99b]. Arithmetic averaging however is a highly non-trivial exercise and one has to rely on either approximations [TW91], partially analytic [GY93] or numerical solutions [Cur94, RS95]. In Ref. [RS95] a simple PDE was derived and bounds on prices of an average price option were derived. These bounds have been improved in Ref. [Tho99].

In this article we provide an alternative approach to derive (partially) analytic solutions of Asian-type contracts with arithmetic averaging. Using the fundamental notion of a local scale invariance [HN99a, HN99b] we derive a general PDE for a (European) Asian-type contract with arithmetic averaging. This result is then linked to the PDE derived by Rogers and Shi [RS95] and we proceed by solving the Laplace-transform of the solution of this PDE for the case of an average strike put, both for the unseasoned and seasoned case. This extends the results of Geman and Yor, which gave the solution for average price options. Next we show that the local scale invariance allows one to identify the average strike call and average price put by substitution of the proper parameters. This is a new result. Finally we consider the problem of a vanilla option on a stock which pays known cash dividends. Again the local scale symmetry allows one to relate the value of such a contract to that of arithmetic average options.

2 Homogeneity and contingent claim pricing

In previous papers [HN99a, HN99b] we have shown that a fundamental property of any properly defined market of tradables[1] is that the price of any claim depending on other tradables in the market should be a homogeneous[2] function of degree one of these same tradables. This property is nothing but a consequence of the simple fact that prices of tradables are only defined with respect to each other. Let us review some of the content of Ref. [HN99a]. Assume that we have a market of $n+1$ basic tradables with prices x_μ ($\mu = 0, \ldots, n$) at time t. The price of any tradable in this market with a payoff depending on the prices of

[1] Tradables are objects which are trivially self-financing: it doesn't cost nor yield money to keep a fixed amount of them. Examples are stocks and bonds. Note that money is not a tradable, unless the interest rate is zero.

[2] A function $f(x_0, \ldots, x_n)$ is called homogeneous of degree r if $f(ax_0, \ldots, ax_n) = a^r f(x_0, \ldots, x_n)$. Homogeneous functions of degree r satisfy the following property (Euler): $\sum_{\mu=0}^{n} x_\mu \frac{\partial}{\partial x_\mu} f(x_0, \ldots, x_n) = r f(x_0, \ldots, x_n)$

these basic tradables should satisfy the following scaling symmetry:

$$V(\lambda x, t) = \lambda V(x, t)$$

which automatically implies[3] (Euler)

$$V(x, t) = x_\mu \partial_{x_\mu} V(x, t)$$

where $\partial/\partial x_\mu \equiv \partial_{x_\mu}$. This is a universal property, independent of the choice of dynamics. We use this fundamental property to derive a general PDE, giving the price of such a claim in a world where the dynamics of the tradables are driven by k independent standard Brownian motions, as follows[4]

$$dx_\mu(t) = x_\mu(t)\big(\sigma_\mu(x, t) \cdot dW(t) + \alpha_\mu(x, t)dt\big), \qquad \text{(no sum)}$$

Consistency requires that both σ_μ and α_μ are homogeneous functions of degree zero in the tradables, i.e. they should only depend on ratios of prices of tradables. Note that we do not specify the numeraire in terms of which the drift and volatility are expressed. This choice is irrelevant for the pricing problem, as we will see. Applying Itô to $V(x, t)$ we get

$$dV(x, t) = \partial_{x_\mu} V(x, t)dx_\mu + \mathcal{L}V(x, t)dt$$

where

$$\mathcal{L}V(x, t) \equiv \left(\partial_t + \frac{1}{2}\sigma_\mu(x, t) \cdot \sigma_\nu(x, t)x_\mu x_\nu \partial_{x_\mu}\partial_{x_\nu}\right)V$$

So, if $V(x, t)$ solves $\mathcal{L}V = 0$ with the payoff at maturity as boundary condition $V(x, T) = f(x)$, then we immediately have a replicating self-financing trading strategy because of the homogeneity property. We will drop the distinction between such derived and basic quantities and always refer to them as tradables. Note that we do not have to use any change of measure to arrive at this result, by keeping the symmetry explicit. Drifts are irrelevant for the derivation of the claim price. Only the requirement of uniqueness of the solution, i.e. no arbitrage, leads to constraints on the drifts terms if deterministic relations exist between the various tradables [HN99a].

2.1 Symmetries of the PDE

The scale invariance of the claim price is inherited by the PDE via an invariance of the solutions of the PDE under a simultaneous shift of all volatility-functions by an arbitrary function $\lambda(x, t)$

$$\sigma_\mu(x, t) \to \sigma_\mu(x, t) - \lambda(x, t) \tag{1}$$

[3]We make use of Einsteins summation convention: repeated indices in products are implicitly summed over, unless stated otherwise.

[4]Both the σ_μ and dW are vectors, the dot denotes an inner-product w.r.t. the k driving diffusions.

Indeed, if V is solves $\mathcal{L}V = 0$, then it also solves

$$\left(\partial_t + \frac{1}{2}(\sigma_\mu(x,t) - \lambda(x,t)) \cdot (\sigma_\nu(x,t) - \lambda(x,t)) x_\mu x_\nu \partial_{x_\mu} \partial_{x_\nu}\right) V = 0$$

This can easily be checked by noting that for homogeneous functions of degree 1 we have

$$x_\mu \partial_{x_\mu} \partial_{x_\nu} V = 0$$

This ensures that terms involving the λ drop out of the PDE. (Note that this equation gives interesting relations between the various Γ's of the claim). From this it follows that V itself must be invariant under the substitution defined by Eq. 1. This corresponds to the freedom of choice of a numeraire. It just states that volatility is a relative concept. Price functions should not depend on the choice of a numeraire.

2.2 The algorithm

To price contingent claims we start out with a basic set of tradables. Using these tradables we may construct new, derived, tradables, whose price-process V depends upon the basic tradables. Of course, these new tradables should be solutions to the basic PDE, $\mathcal{L}V = 0$. Their payoff functions serve as boundary conditions. (Note that prices of basic tradables trivially satisfy the PDE, by construction). If the derived tradables are constructed in this way, we can use them just like any other tradable. In particular, we can use them as underlying tradables, in terms of which the price of yet other derivative claims can be expressed (and so on...) In fact, this is a fundamental property that any correctly defined market should posses. It amounts to a proper choice of coordinates to describe the economy.

The general approach to the pricing of a path-dependent claim in our formalism can be described as follows.

1. The payoff is written in terms of tradable objects.
2. A PDE is derived for the claim price with respect to these tradables.
3. The PDE is solved.
4. Possible consistency check: the solution should be invariant under the substitution Eq. 1 (numeraire independence).

2.3 Generalized put-call symmetries

As an example of the strength of this symmetry, and to show the natural embedding in our formalism, consider an economy with two tradables with prices denoted by $x_{1,2}$ and dynamics given by $(i = 1, 2)$

$$dx_i(t) = x_i(t)\sigma_i(x_1, x_2, t) \cdot dW(t) + \ldots \qquad \text{(no sum)}$$

It is easy to see that under certain conditions there should be a generalized put-call symmetry. Any claim with payoff $f(x_1, x_2)$ at maturity and price $V(x_1, x_2, t)$ should satisfy

$$\left(\partial_t + \frac{1}{2}|\sigma(x_1, x_2, t)|^2 x_1^2 \partial_{x_1}^2\right) V = 0$$

where $\sigma(x_1, x_2, t) \equiv \sigma_1(x_1, x_2, t) - \sigma_2(x_1, x_2, t)$. Homogeneity implies that it also solves

$$\left(\partial_t + \frac{1}{2}|\sigma(x_1, x_2, t)|^2 x_2^2 \partial_{x_2}^2\right) V = 0$$

Therefore, if $|\sigma(x_1, x_2, t)|^2 = |\sigma(x_2, x_1, t)|^2$, this PDE can be rewritten as

$$\left(\partial_t + \frac{1}{2}|\sigma(x_2, x_1, t)|^2 x_2^2 \partial_{x_2}^2\right) V = 0$$

and we see that $V(x_2, x_1, t)$ with payoff $f(x_2, x_1)$ is a solution, too. This is nothing but a generalized put-call symmetry. In the first case x_2 acts as numeraire, in the second case x_1 takes over this role. The usual put-call symmetry follows if we take a constant σ and let x_1, x_2 represent a stock and a bond respectively.

2.4 Lognormal asset prices

In an economy with lognormal distributed asset-prices

$$dx_\mu(t) = x_\mu(t)\sigma_\mu(t) \cdot dW(t) + \ldots \qquad \text{(no sum)}$$

it is possible to write down a very elegant formula for European-type claims, as was shown in Ref. [HN99a]

$$V(x_0, \ldots, x_n, t) = \int V(x_0\phi(z - \theta_0), \ldots, x_n\phi(z - \theta_n), T) d^m z \tag{2}$$

with

$$\phi(z) = \frac{1}{\left(\sqrt{2\pi}\right)^m} \exp\left(-\frac{1}{2}\sum_{i=1}^m z_i^2\right)$$

The θ_μ are m-dimensional vectors, which follow from a singular value decomposition of the covariance matrix $\Sigma_{\mu\nu}$ of rank $m \leq k$:

$$\Sigma_{\mu\nu} \equiv \int_t^T \sigma_\mu(u) \cdot \sigma_\nu(u) du = \theta_\mu \cdot \theta_\nu$$

3 Arithmetic Asians

In this section we will consider the pricing of Arithmetic Asian options. Since this is the only type of Asian options that we will look at, we will omit the word 'Arithmetic' in the sequel. Note that parts of this material already appeared in Ref. [HN99b]. A fundamental building block in the construction of a European Asian option, expiring at time T, is a tradable which at time T represents the value of a stock at an earlier time s. But to define this, we must first agree how to translate value through time. For this, we need a reference asset. A convenient choice is to take a bond $P(t,T)$ (or $P(t)$ for short), which matures at time T, as reference. Then we can define

$$Y_s(t) = \begin{cases} S(t) & t < s \\ \frac{S(s)}{P(s)} P(t) & t \geq s \end{cases}$$

In words, this is a portfolio where one starts out with a stock, and converts it into a bond at time $t = s$. It is trivially self-financing. To set the stage, we will assume that the interest rate has constant value r, as is usual in the Black-Scholes context (stochastic interest rates are much harder to handle, see Ref. [HN99b]). In that case the bond with maturity T has value $e^{-r(T-t)}$ when expressed in the currency in which it is nominated (say dollars). Consequently, an amount $e^{-r(T-s)}$ of the tradable Y_s will have a dollar value at time T which is equal to the dollar value of the stock at time s. We will also assume that the contracts are initiated at time $t = 0$, unless stated otherwise. Note that for $t \in [0, T]$ we have

$$\begin{aligned} S(t) &= Y_T(t) \\ P(t) &= \frac{P(0)}{S(0)} Y_0(t) \end{aligned} \tag{3}$$

3.1 Discretely sampled Asians

The main reason for introducing the objects $Y_s(t)$ is that they constitute a natural basis of tradables in which prices and payoffs of arithmetic Asians can be expressed. For example, the payoff of a discretely sampled average price call (APC) can be written as

$$\left(\sum_i w(t_i) Y_{t_i}(T) - K P(T) \right)^+ = \left(\left(\sum_i w(t_i) \frac{S(t_i)}{P(t_i)} - K \right) P(T) \right)^+$$

where $\{t_i\}$ is a set of sample times, $w(t_i)$ are corresponding weights, and K is the strike. We use the notation $(\cdot)^+$ for $\max(\cdot, 0)$. Observe that the payoff explicitly contains $P(T) = 1\$$ to make it homogeneous of degree one in the tradables. In a similar way, the payoff of an average strike put (ASP) becomes

$$\left(\sum_i w(t_i) Y_{t_i}(T) - k S(T) \right)^+$$

where k is a (generalized) strike. In view of Eqs. 3, we see that both options are in fact instances of a more general discrete Asian option, which is defined by the payoff

$$V(\{t_i\}, w, T) = \left(\sum_i w(t_i) Y_{t_i}(T) \right)^+ \tag{4}$$

3.2 Valuation by multiple integrals

In this section we want to calculate the value of the generalized option defined by Eq. 4, at the time the contract is initiated, i.e. $t = 0$. This value is known as its *unseasoned* value. Without loss of generality, we will assume that there are $N + 1$ sample times, satisfying $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$. If we take the bond as numeraire, and assume that the stock price follows a lognormal price process

$$dS(t) = \sigma S(t) dW(t) + \cdots$$

then it is easy to derive that the tradables $Y_s(t)$ satisfy

$$dY_s(t) = \mathbf{1}_{t<s} \sigma Y_s(t) dW(t) + \cdots$$

where $\mathbf{1}_{t<s}$ is the indicator function. It equals one when $t < s$ and zero otherwise. So the $Y_s(t)$ also follow lognormal price processes with a time dependent volatility, and we can directly use the results of section 2.4. The first step is to calculate the variance-covariance matrix of the tradables

$$\Sigma_{ij} = \sigma^2 \int_0^T \mathbf{1}_{t<t_i} \mathbf{1}_{t<t_j} dt = \sigma^2 \min(t_i, t_j)$$

The rank of this matrix is N. A singular value decomposition $\Sigma_{ij} = \theta_i \cdot \theta_j$ is given by

$$\theta_0 = (\underbrace{0, \ldots, 0}_{N}), \qquad \theta_i = (\beta_1, \ldots, \beta_i, \underbrace{0, \ldots, 0}_{N-i})$$

and the β_i are defined by

$$\beta_i = \sigma \sqrt{t_i - t_{i-1}}$$

The value of the unseasoned option can now be written as an N-dimensional integral (here $z = (z_1, \ldots, z_N)$)

$$V(\{t_i\}, w, 0) = S(0) \int d^N z \left(\sum_{i=0}^{N} w(t_i) \phi(z - \theta_i) \right)^+ \tag{5}$$

where we used the fact that $Y_s(0) = S(0)$ for all $0 \le s \le T$. This is in fact a Feynman-Kac formula.

3.3 An interesting duality

In this section, we will again calculate the unseasoned value of the option in Eq. 4, but this time we take the stock as numeraire. This corresponds to a shift of σ in all volatility functions. We find that $Y_s(t)$ now satisfies

$$dY_s(t) = \mathbf{1}_{t>s}\sigma Y_s(t)dW(t) + \cdots$$

In this case the variance-covariance matrix becomes

$$\hat{\Sigma}_{ij} = \sigma^2 \int_0^T \mathbf{1}_{t>t_i}\mathbf{1}_{t>t_j}dt = \sigma^2 \min(T - t_i, T - t_j)$$

and a singular value decomposition $\hat{\Sigma}_{ij} = \hat{\theta}_i \cdot \hat{\theta}_j$ is given by

$$\hat{\theta}_i = (\underbrace{0, \ldots, 0}_{i}, \beta_{i+1}, \ldots, \beta_N), \qquad \hat{\theta}_N = (\underbrace{0, \ldots, 0}_{N})$$

So an alternative expression for the price of the option is

$$V(\{t_i\}, w, 0) = S(0) \int d^N z \left(\sum_{i=0}^{N} w(t_i)\phi(z - \hat{\theta}_i) \right)^+$$

However, if we compare this result with Eq. 5, we see that it could also be interpreted as the value of an option with payoff

$$\left(\sum_{i=0}^{N} w(t_i) Y_{T-t_i}(T) \right)^+$$

In other words, two options which are related by the following substitution in their payoff

$$\boxed{Y_t(T) \leftrightarrow Y_{T-t}(T)} \qquad (6)$$

have the same value at $t = 0$. We will call this T-duality (T from Time-reversal). It is a very interesting symmetry operation because, in view of Eqs. 3, we can use it to relate the values of unseasoned average strike and average price options. We will come back to this point in section 3.6. Note that Eq. 6 takes a simple form by virtue of the fact that we are working in a basis of tradables.

3.4 A PDE approach

In this section we consider a PDE approach to the pricing of Asian options. We derive a very general PDE, which can be related to the one that is usually found in the literature. Our PDE, however, has the advantage of being manifestly numeraire independent by virtue of the fact that it is expressed in a basis of tradables. It can be used to price both American and European style options,

but we will focus on the European case here. We will come back to American Asians in future work. The basic idea in the derivation of the PDE is, instead of introducing a tradable for each sample date, to introduce one new tradable $\bar{S}(t)$, which is a weighted sum over $Y_s(t)$ (obviously, a sum of tradables is again a tradable). This allows us to consider continuously sampled Asians in a proper way. Also

$$\bar{S}(t) = \int_0^T w(s) Y_s(t) ds \equiv \phi(t) S(t) + A(t) P(t)$$

where $A(t)$ is proportional to the running average

$$A(t) = \int_0^t w(s) \frac{S(s)}{P(s)} ds$$

and

$$\phi(t) = \int_t^T w(s) ds \qquad \longleftrightarrow \qquad w(t) = -\partial_t \phi(t), \quad \phi(T) = 0$$

Of course this approach also incorporates discretely sampled Asians. In that case, $w(t)$ will be a sum of Dirac delta-functions and $\phi(t)$ will be a piecewise constant function, making jumps at sample dates. If we choose the bond $P(t)$ as numeraire, and assume that $S(t)$ satisfies

$$dS(t) = \sigma S(t) dW(t) + \cdots$$

then it is obvious that the new tradable $\bar{S}(t)$ satisfies

$$d\bar{S}(t) = \phi(t) \sigma S(t) dW(t) + \cdots$$

This straightforwardly leads to the following PDE for options which depend on $\bar{S}, S$ and P

$$\left(\partial_t + \frac{1}{2} \sigma^2 S^2 (\partial_S + \phi \partial_{\bar{S}})^2 \right) V = 0 \tag{7}$$

If we perform a change of variables in the PDE, eliminating $\bar{S}$ in favor of A, we find

$$\left(\partial_t + w \frac{S}{P} \partial_A + \frac{1}{2} \sigma^2 S^2 \partial_S^2 \right) V = 0$$

which is closer to the usual formulation. However, since $A(t)$ does not correspond to the value of a tradable object, this form of the PDE looses the manifest symmetry.

3.5 Analytical solutions

In this section we derive a Laplace transform representation for prices of average strike options in the case that sampling is continuous with an exponential weight function. This is in fact the counterpart of the calculation of Geman and

Yor [GY93] for the price of average price options, although they used entirely different methods to derive it. The results for average strike options turn out to be somewhat more involved, mainly because there is no simple relation between prices of seasoned and unseasoned options, while a simple relation does exist in the case of average price options, as we will see. Now, by definition, the payoff of an average strike option is defined in terms of $\bar{S}$ and S only, P does not appear. For example, the payoff of an ASP is given by

$$(\bar{S}(T) - kS(T))^+ \tag{8}$$

Because of this fact, it is natural to choose S as numeraire. Dropping the derivative w.r.t. P, the PDE then reduces to

$$\left(\partial_t + \frac{1}{2}\sigma^2(\bar{S} - \phi S)^2 \partial_{\bar{S}}^2\right) V = 0$$

Next, introduce $x \equiv \bar{S}/S$ and set $\hat{V}(x,t) \equiv V/S$. This reduces the dimension of the PDE by one

$$\left(\partial_t + \frac{1}{2}\sigma^2(x - \phi)^2 \partial_x^2\right) \hat{V} = 0$$

The resulting PDE is closely related to the one found by Rogers and Shi [RS95]. In fact, they can be transformed into each other by a variable change $y = x - \phi$. But our derivation of the PDE seems to be more natural: working in a basis of tradables guides us in the right direction. At this point, we will make the specific choice of an exponential weight function

$$w(t) = \frac{e^{-\gamma(T-t)}}{T}$$

Remember that if the interest rate is constant and equal to r, then the choice $\gamma = r$ leads to an equally weighted average in terms of the dollar price of the stock. We now find

$$\phi(t) = \frac{1 - e^{-\gamma(T-t)}}{\gamma T}$$

A change of variables is now in place

$$\tau \equiv T - t, \qquad z \equiv \frac{2e^{-\gamma\tau}}{\sigma^2 T(x - \phi)} = \frac{2Se^{-\gamma\tau}}{\sigma^2 TAP}, \qquad s \equiv \sigma^2\tau, \qquad \kappa \equiv \frac{\gamma}{\sigma^2}$$

This transforms the PDE to

$$\left(-\partial_s + \left((\kappa+1)z - \frac{1}{2}z^2\right)\partial_z + \frac{1}{2}z^2\partial_z^2\right) \hat{V} = 0$$

A Laplace-transform with respect to s yields

$$\left(-\lambda + \left((\kappa+1)z - \frac{1}{2}z^2\right)\partial_z + \frac{1}{2}z^2\partial_z^2\right) u = -f(z)$$

where $u(z,\lambda)$ is the transformed function, and $f(z)=\hat{V}(z,T)$ denotes the payoff at maturity. Next, let us define

$$u \equiv e^{\frac{1}{2}z} z^{-\kappa-1} w$$

Then w satisfies

$$\left(\partial_z^2 - \frac{1}{4} + \frac{(\kappa+1)}{z} + \frac{(\frac{1}{4}-\mu^2)}{z^2}\right) w = -2e^{-\frac{1}{2}z} z^{\kappa-1} f(z) \tag{9}$$

where we introduced

$$\mu \equiv \sqrt{\left(\kappa+\tfrac{1}{2}\right)^2 + 2\lambda}$$

The homogeneous part of Eq. 9 is Whittaker's equation, and its solutions are the Whittaker functions $M_{\kappa+1,\mu}(z)$ and $W_{\kappa+1,\mu}(z)$ (see appendix). To solve Eq. 9 we will make use of a Green's function approach. A Green's function with proper behaviour at the boundaries is given by

$$G(x,y) = \frac{1}{Q}\left(M_{\kappa+1,\mu}(x) W_{\kappa+1,\mu}(y) \mathbf{1}_{x<y} + W_{\kappa+1,\mu}(x) M_{\kappa+1,\mu}(y) \mathbf{1}_{x>y}\right)$$

where Q is the Wronskian of the two solutions

$$Q = W_{\kappa+1,\mu}(z)\partial_z M_{\kappa+1,\mu}(z) - M_{\kappa+1,\mu}(z)\partial_z W_{\kappa+1,\mu}(z) = \frac{\Gamma(1+2\mu)}{\Gamma(-\frac{1}{2}-\kappa+\mu)}$$

In terms of this, the solution can be written as

$$u(z,\lambda) = 2e^{\frac{1}{2}z} z^{-\kappa-1} \int_0^\infty G(x,z) e^{-\frac{1}{2}x} x^{\kappa-1} f(x) dx$$

The ASP payoff defined in Eq. 8 corresponds to the choice

$$f(z) = \left(\frac{2}{\sigma^2 T z} - k\right)^+ = k\left(\frac{a}{z}-1\right)^+, \qquad a \equiv \frac{2}{\sigma^2 k T}$$

Inserting this in the integral, we find that the solution falls apart in two ranges, $z<a$ and $z \geq a$ (or, equivalently, $e^{-\gamma\tau}kS < AP$ and $e^{-\gamma\tau}kS \geq AP$). For details of this calculation we refer to the appendix. In the former case, we find

$$\begin{aligned} u(z,\lambda) = {} & \frac{2ke^{\frac{z-a}{2}} z^{-\kappa-1} a^\kappa \Gamma(-\frac{1}{2}-\kappa+\mu)}{\Gamma(1+2\mu)} W_{\kappa-1,\mu}(a) M_{\kappa+1,\mu}(z) \\ & + \frac{2k\Big(z\big((\kappa-\frac{1}{2})^2-\mu^2\big) + a\big(z-\big((\kappa+\frac{1}{2})^2-\mu^2\big)\big)\Big)}{z\big((\kappa-\frac{1}{2})^2-\mu^2\big)\big((\kappa+\frac{1}{2})^2-\mu^2\big)} \end{aligned}$$

The second term is exactly the Laplace transform of $x-k$. In the latter case we find

$$u(z,\lambda) = \frac{2ke^{\frac{z-a}{2}} z^{-\kappa-1} a^\kappa \Gamma(-\frac{1}{2}-\kappa+\mu)}{(-\frac{1}{2}+\kappa+\mu)(\frac{1}{2}+\kappa+\mu)\Gamma(1+2\mu)} M_{\kappa-1,\mu}(a) W_{\kappa+1,\mu}(z)$$

Therefore, the solution can be written as

$$V_{\mathrm{ASP}}(k,\gamma,t,T)=\begin{cases} kS(t)I_1+\bar{S}(t)-kS(t) & e^{-\gamma\tau}kS<AP \\ kS(t)I_2 & e^{-\gamma\tau}kS\geq AP \end{cases}$$

where I_1 and I_2 are defined by inverse Laplace-transforms

$$I_1 \equiv \frac{e^{\frac{z-a}{2}}z^{-\kappa-1}a^{\kappa}}{\pi i}\int_{\rho-i\infty}^{\rho+i\infty}\frac{\Gamma(-\frac{1}{2}-\kappa+\mu)W_{\kappa-1,\mu}(a)M_{\kappa+1,\mu}(z)e^{\lambda s}}{\Gamma(1+2\mu)}d\lambda$$

$$I_2 \equiv \frac{e^{\frac{z-a}{2}}z^{-\kappa-1}a^{\kappa}}{\pi i}\int_{\rho-i\infty}^{\rho+i\infty}\frac{\Gamma(-\frac{1}{2}-\kappa+\mu)M_{\kappa-1,\mu}(a)W_{\kappa+1,\mu}(z)e^{\lambda s}}{(-\frac{1}{2}+\kappa+\mu)(\frac{1}{2}+\kappa+\mu)\Gamma(1+2\mu)}d\lambda$$

where ρ is an arbitrary constant chosen so that the contour of integration lies to the right of all singularities in the integrand. These integrals can be evaluated numerically [AW95, Sha98]. Note that the value of an average strike call (ASC) follows simply from put-call parity, that is, we use

$$\Big(\bar{S}(T)-kS(T)\Big)^{+}-\Big(kS(T)-\bar{S}(T)\Big)^{+}=\bar{S}(T)-kS(T)$$

and find

$$V_{\mathrm{ASC}}(k,\gamma,t,T)=\begin{cases} kS(t)I_1 & e^{-\gamma\tau}kS<AP \\ kS(t)I_2-\bar{S}(t)+kS(t) & e^{-\gamma\tau}kS\geq AP \end{cases}$$

The expression for the value of the ASP at $t=0$, i.e. its unseasoned value, simplifies considerably. In this case $z\to\infty$ and $s=\sigma^2T=2/(ak)$, and we find

$$V_{\mathrm{ASP}}(k,\gamma,0,T)=kS(0)\frac{e^{-\frac{a}{2}}a^{\kappa}}{\pi i}\int_{\rho-i\infty}^{\rho+i\infty}\frac{\Gamma(-\frac{1}{2}-\kappa+\mu)M_{\kappa-1,\mu}(a)\exp(\frac{2\lambda}{ak})}{(-\frac{1}{2}+\kappa+\mu)(\frac{1}{2}+\kappa+\mu)\Gamma(1+2\mu)}d\lambda$$

3.6 T-duality and average price options

In this section we will use the T-duality, found in section 3.3, to relate prices of unseasoned average strike and average price options. We will focus on the case where the weight function is exponential. To indicate the weight function used in the definition of $\bar{S}$, we use a subscript γ, i.e. we define

$$\bar{S}_{\gamma}(t)=\frac{1}{T}\int_0^T e^{-\gamma(T-s)}Y_s(t)ds$$

It is a straightforward calculation to see that the action of the duality Eq. 6 (remark: we use a continuum limit of this result) on the tradables $\bar{S}_{\gamma}$, S and P is given by

$$\boxed{\begin{aligned}\bar{S}_{\gamma}(T)&\leftrightarrow e^{-\gamma T}\bar{S}_{-\gamma}(T)\\ S(T)&\leftrightarrow\frac{S(0)}{P(0)}P(T)\end{aligned}}$$

Applying this to the payoff of an average price call gives

$$(\bar{S}_\gamma(T) - KP(T))^+ \leftrightarrow e^{-\gamma T}\left(\bar{S}_{-\gamma}(T) - \frac{KP(0)}{e^{-\gamma T}S(0)}S(T)\right)^+$$

i.e., it transforms it into the payoff of an ASP. Therefore we see that the unseasoned value of an APC can be expressed as

$$V_{\mathrm{APC}}(K, \gamma, 0, T) = e^{-\gamma T} V_{\mathrm{ASP}}\left(\frac{KP(0)}{e^{-\gamma T}S(0)}, -\gamma, 0, T\right)$$

Similarly, we obtain the value of an unseasoned average price put (APP)

$$V_{\mathrm{APP}}(K, \gamma, 0, T) = e^{-\gamma T} V_{\mathrm{ASC}}\left(\frac{KP(0)}{e^{-\gamma T}S(0)}, -\gamma, 0, T\right)$$

In this way, we actually reproduce the well known results by Geman and Yor [GY93].

3.7 On seasoned Asians

It is a well-known fact that the price of a seasoned average price option can be expressed in terms of the price of an unseasoned average price option with a different strike. Let us look at the mechanism behind this. We consider an exponentially weighted, continuously sampled Asian with a total lifetime of M, expiring at time T (so it is initiated at time $T - M < 0$) and we are interested in its price at $t = 0$. As before, the payoff of such an option can be expressed in terms of tradables S, P and

$$\bar{S}_{\gamma,M}(t) = \frac{1}{M}\int_{T-M}^{T} e^{-\gamma(T-s)} Y_s(t) ds$$

where we explicitly show the longer sample period in the definition by the subscript M. Now if $t \in [0, T]$ we can write

$$\bar{S}_{\gamma,M}(t) = \frac{T\bar{S}_\gamma(t)}{M} + \frac{P(t)}{M}\int_{T-M}^{0} e^{-\gamma(T-s)}\frac{S(s)}{P(s)}ds \equiv \frac{T\bar{S}_\gamma(t) + AP(t)}{M} \qquad (10)$$

where A is proportional to the average over the time period up to $t = 0$. Substituting this in the payoff of an APC, we get

$$\left(\bar{S}_{\gamma,M}(T) - KP(T)\right)^+ = \frac{T}{M}\left(\bar{S}_\gamma(T) - \hat{K}P(T)\right)^+$$

with

$$\hat{K} = \frac{MK - A}{T}$$

This shows that the value of the *seasoned* APC that we are considering can be expressed in terms of the value of an *unseasoned* APC with a modified strike as

$$\frac{T}{M} V_{\mathrm{APC}}(\hat{K}, \gamma, 0, T)$$

Of course, the same trick also works for average price puts. Note that it is possible for the strike $\hat{K}$ to become negative. In that case, the option becomes trivial. One might wonder what happens if we substitute Eq. 10 in the payoff of an ASC. It turns out that in that case, things do not combine in a nice way. Indeed, we find

$$\left(kS(T) - \bar{S}_{\gamma,M}(T)\right)^+ = \left(\frac{MkS(t) - T\bar{S}_\gamma(T) - AP(T)}{M}\right)^+ \tag{11}$$

Fortunately, we already have an expression for the value of a seasoned ASC. So we can use the formula in reverse, to price options with a payoff given by the RHS of Eq. 11. This will turn out to be useful in the next section.

4 Cash-dividend

It well known in the literature how to price options on a stock paying a known dividend yield. However, in many cases it is more realistic to assume that the cash amount of the dividend rather than the yield is known in advance. This makes the pricing problem considerably harder. In this section we show that the problem is equivalent to the pricing of Asian options. In fact, we show that their prices are connected by the put-call symmetries of section 2.3. This allows us to use all the techniques for the valuation of Asian options in the context of options involving cash-dividend. The setting is as follows. We assume that the stock follows geometric Brownian motion between dividend payments, with fixed volatility σ (taking the bond as numeraire). Dividends are paid at a set of discrete times $\{t_i\}$, $i = 1, 2, 3, \ldots$, $0 < t_1 < t_2 < \cdots$ and are expressed in units of the bond $\delta(t_i)P$. Since we will be interested in options with maturity T, we will use a bond with this same maturity. We assume that the interest rate is fixed and equal to r, so the bond has dollar value $e^{-r(T-t)}$. By $S_i(t)$ we mean the price of the stock between t_i and t_{i+1}, in other words

$$S(t) = S_i(t), \qquad \text{for } t_i \leq t < t_{i+1}$$

So at $t = t_1$ a portfolio consisting of 1 stock becomes

$$S_0 \to \delta(t_1)P + S_1$$

In order to avoid arbitrage, we assume that the left-hand side equals the right-hand side at t_1. This can be used to extend the definition of S_0 to all $t < t_2$ as follows

$$S(t) = S_1(t) = \left(1 - \frac{\delta(t_1)P(t_1)}{S_0(t_1)}\right) S_0(t), \qquad t_1 \leq t < t_2$$

Note that S_0, by construction, does *not* make a jump at t_1. In fact, S_0 corresponds to the value of the self-financing portfolio that one gets by directly reinvesting the cash-dividend payment into the stock again. So S_0 is a tradable object. We can repeat the process for the dividend payment at t_2

$$S_1 \to \delta(t_2)P + S_2$$

Again, S_2 can be expressed in terms of S_0, extending the definition of S_0 to all $t < t_3$

$$S(t) = S_2(t) = \left(1 - \frac{\delta(t_2)P(t_2)}{S_1(t_2)}\right) S_1(t) =$$

$$\left(1 - \frac{\delta(t_1)P(t_1)}{S_0(t_1)} - \frac{\delta(t_2)P(t_2)}{S_0(t_2)}\right) S_0(t), \qquad t_2 \leq t < t_3$$

By repeating this process, we find that the value of a portfolio V which we get by starting with one stock at $t = 0$ and holding it, together with all its cumulative dividends up to time t (note that this portfolio is also a tradable object, while the stock by itself is not) is given by

$$V(t) = S(t) + \sum_{t_i \leq t} \delta(t_i)P(t) = \left(1 - \sum_{t_i \leq t} \frac{\delta(t_i)P(t_i)}{S_0(t_i)}\right) S_0(t) + \sum_{t_i \leq t} \delta(t_i)P(t)$$

where S_0 just follows a lognormal price process. Now if we consider a European option of the stock with maturity T, we can define the cumulative dividends up to maturity as follows

$$C(t) \equiv \sum_{t_i \leq T} \delta(t_i)P(t)$$

Using this we can write

$$V(t) = S_0(t) + C(t) - \bar{P}(t), \qquad t \leq T$$

where

$$\bar{P}(t) \equiv \sum_{t < t_i \leq T} \delta(t_i)P(t) + \sum_{t_i \leq t} \frac{\delta(t_i)P(t_i)}{S_0(t_i)} S_0(t)$$

In terms of these new tradables, we can write

$$S(T) = S_0(T) - \bar{P}(T)$$

Now the connection with Asian option becomes clear. They can be transformed into each other by the exchange of S_0 and P, i.e. by using put-call symmetry. In fact, we can introduce tradable objects, similar to the $Y_s(t)$, as follows

$$X_s(t) = \begin{cases} P(t) & t \leq s \\ \frac{P(s)}{S_0(s)} S_0(t) & t \geq s \end{cases}$$

In terms of these, we can write

$$\bar{P}(t) = \sum_{t_i \leq T} \delta(t_i) X_{t_i}(t)$$

Again, for $t \in [0, T]$, we have

$$P(t) = X_T(t)$$

$$S_0(t) = \frac{S(0)}{P(0)} X_0(t)$$

Therefore we see that the payoff of a plain vanilla option on a stock paying cash dividends takes the general form

$$\left(\sum_i \delta(t_i) X_{t_i}(T)\right)^+$$

with certain weights δ. Taking $S_0(t)$ as numeraire, we see that the tradables $X_s(t)$ satisfy

$$dX_s(t) = \mathbf{1}_{t<s} \sigma X_s(t) dW(t) + \cdots$$

and we can use the integral approach described in section 3.2 to price the option. Alternatively we can use a PDE approach, i.e. we generalize the definition of $\bar{P}$ as follows, cf. the steps in 3.4,

$$\bar{P}(t) = \int_0^T \delta(s) X_s(t) ds = \phi(t) P(t) + A(t) S_0(t)$$

A PDE which describes the price process of an option depending on S_0, P and $\bar{P}$ (this class includes plain vanilla options on stocks paying cash-dividends) can now easily be derived (just take the stock as numeraire)

$$\left(\partial_t + \frac{1}{2}\sigma^2 P^2 (\partial_P + \phi \partial_{\bar{P}})^2\right) V = 0$$

It is instructive to consider a call option on a stock which pays a continuous stream of cash dividends, with exponential weights. One can use this as an approximation to the value of an option where the underlying stock pays a long stream of discrete cash dividends. So let us define

$$\bar{P}_\gamma(t) \equiv \frac{1}{T} \int_0^T e^{-\gamma(T-s)} X_s(t) ds$$

The natural choice is $\gamma = -r$, which corresponds to a constant dividend stream in terms of dollars. The payoff of a plain vanilla call becomes

$$\left(S(T) - KP(T)\right)^+ \to \left(S_0 - \delta T \bar{P}_{-r}(T) - KP(T)\right)^+$$

where δ parametrizes the dividend stream. We will from now on omit the subscript from the S_0. By exchanging S and P, exploiting put-call symmetry, we get

$$\left(P(T) - \delta T \bar{S}_{-r}(T) - KS(T)\right)^+$$

which is the payoff of some Asian option. In fact, by using T-duality, the payoff can be related to one that corresponds to a seasoned average strike call (see Eq. 11)

$$\left(\frac{P(0)}{S(0)} S(T) - \delta T e^{rT} \bar{S}_r(T) - \frac{S(0)}{P(0)} KP(T)\right)^+$$

and we can use the analytical results that we derived for this type of option to write down the price of this instrument.

5 Conclusion and outlook

In this article we have shown the power of symmetries to derive prices of complex exotic options. We focused on arithmetic average options in a Black-Scholes setting. By choosing an appropriate basis of tradables, i.e. self-financing portfolios, it becomes a straightforward matter to write down the governing PDE for the option price. We then proceed to derive the Laplace-transformed price of a European average strike option. This result extends the result of Geman and Yor [GY93] for the average price option. Next we show the power of the underlying symmetry, by showing the equivalence of the unseasoned arithmetic average strike and price options after a suitable transformation of parameters. Seasoned options can be treated in a similar way. Finally we exploit the symmetry in the problem to show that vanilla options on stocks paying cash dividends are equivalent, after suitable transformations, to arithmetic Asian options, thus providing a method to price these type of options.

Let us remark that the present discussion carries over without too much changes to the case of basket options and swaptions. We will discuss this in future work. Also we did not discuss the case of an arithmetic Asian option with early-exercise features. This is however simple to implement and we will come back to this in a future work.

A Whittaker functions

In this appendix we enumerate some useful properties of Whittaker functions. More information can be found in e.g. [AS64, PBM86]. The Whittaker functions $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ are solutions to Whittaker's PDE

$$\left(\partial_z^2 - \frac{1}{4} + \frac{\kappa}{z} + \frac{(\frac{1}{4} - \mu^2)}{z^2}\right) f = 0$$

These functions are defined as

$$M_{\kappa,\mu}(z) = e^{-\frac{1}{2}z} z^{\mu+\frac{1}{2}} {}_1\mathrm{F}_1(\tfrac{1}{2} + \mu - \kappa, 1 + 2\mu, z)$$

where the confluent hypergeometric function is given by

$${}_1\mathrm{F}_1(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n) z^n}{\Gamma(b+n) n!}$$

and

$$W_{\kappa,\mu}(z) = e^{-\frac{1}{2}z} z^{\mu+\frac{1}{2}} \Psi(\tfrac{1}{2} + \mu - \kappa, 1 + 2\mu, z)$$

where Ψ is the Tricomi function

$$\Psi(a, b, z) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-zt} t^{a-1} (1+t)^{b-a-1} dt$$

There are the following interesting relations

$$W_{\kappa,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\kappa)} M_{\kappa,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\kappa)} M_{\kappa,-\mu}(z)$$

$$z^{-\mu-\frac{1}{2}} M_{\kappa,\mu}(z) = (-z)^{-\mu-\frac{1}{2}} M_{-\kappa,\mu}(-z)$$

To evaluate the price of the ASP we made use of the following definite integrals

$$\begin{aligned}
&\int e^{-\frac{1}{2}z} z^{\kappa-1} W_{\kappa+1,\mu}(z) dz = -e^{-\frac{1}{2}z} z^{\kappa} W_{\kappa,\mu}(z) \\
&\int e^{-\frac{1}{2}z} z^{\kappa-2} W_{\kappa+1,\mu}(z) dz = e^{-\frac{1}{2}z} z^{\kappa-1} (W_{\kappa-1,\mu}(z) - W_{\kappa,\mu}(z)) \\
&\int e^{-\frac{1}{2}z} z^{\kappa-1} M_{\kappa+1,\mu}(z) dz = \frac{e^{-\frac{1}{2}z} z^{\kappa}}{\Gamma(\frac{3}{2}+\kappa+\mu)} \Gamma(\tfrac{1}{2}+\kappa+\mu) M_{\kappa,\mu}(z) \\
&\int e^{-\frac{1}{2}z} z^{\kappa-2} M_{\kappa+1,\mu}(z) dz = \frac{e^{-\frac{1}{2}z} z^{\kappa-1}}{\Gamma(\frac{3}{2}+\kappa+\mu)} \times \\
&\qquad \times (\Gamma(\tfrac{1}{2}+\kappa+\mu) M_{\kappa,\mu}(z) + \Gamma(-\tfrac{1}{2}+\kappa+\mu) M_{\kappa-1,\mu}(z))
\end{aligned}$$

To calculate its unseasoned value, we recall the asymptotic behaviour of $W_{\kappa,\mu}(z)$

$$W_{\kappa,\mu}(z) \sim e^{-\frac{1}{2}z} z^{\kappa}, \qquad z \to \infty$$

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